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Exam – November, 19 2018

Many questions can be answered independently.

Part 1

Exercise 1: Free fall

We consider a particle of mass m in a gravitation field $\vec{g} = -g\vec{e}_z$. The particle is launched from the ground $z = 0$ with an initial speed $\vec{v}_0 = v_0\vec{e}_z$.

1. Write the Lagrangian $L(t, z, \dot{z})$. Give the expression of p, conjugate variable of z. Deduce the expression of the Hamiltonian $H(z, p)$.

2. We define $P = H(z, p)$. Suggest a function $Z(z, p)$ such that the transformation $(z, p) \to (Z, P)$ is canonical.

3. Write Hamilton equations, find the expression of $Z(t)$ and $P(t)$, then give $z(t)$.

Exercise 2: Vibrations of a thin blade

We consider a uniform, homogeneous thin slice of linear density μ, of length ℓ, initially at rest horizontally. We study the vibrations of this blade when deformed (we will neglect the influence of gravity).

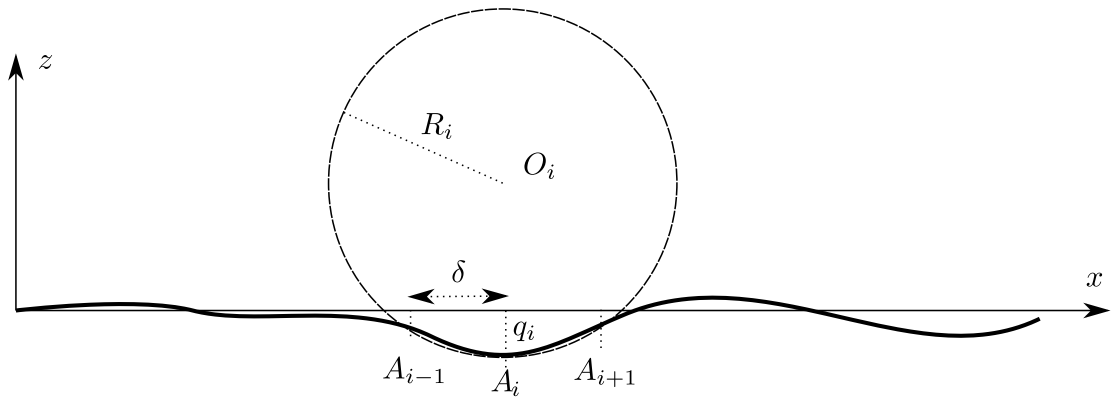

Figure 1: Discrete model of the blade deformation. The circle of center O_i and radius R_i go through A_i, A_{i-1} and A_{i+1}. q_i is the vertical deviation of A_i relative to the horizontal axis. With small deformations, $q_i \ll R_i$.

Discrete model We separate the blade in N segments of length δ. The dynamics of the blade is vertical. For each segment, let A_i be its center of gravity, and q_i the vertical deviation of A_i relative to equilibrium. We only consider small deformations, δ being thus approximatively the length of the x projection of the segment $[A_i A_{i+1}]$.

4. Give the kinetic energy of the blade.

5. Show that the local curvature radius R_i is given by

$$\frac{1}{R_i} = \frac{|2q_i - q_{i-1} - q_{i+1}|}{\delta^2}. \tag{1}$$

One could use that R_i is the radius of the circle going through A_i, A_{i-1} and A_{i+1}. One reminds the equation of a circle of center O of coordinates (X_O, Z_O) and of radius R: $(x - X_O)^2 + (z - Z_O)^2 = R^2$. Terms in q^2 will be neglected compared to q terms.

6. The potential energy by unit length associated to a local deformation of the blade is $EI/(2R^2)$ where E is the Young's modulus, $I = se^2/12$ with e the thickness, s the transversal section, and R the local curvature radius. Deduce the expression of the Lagrangian $L(\{q_i, \dot{q}_i\}_{1\leq i\leq N})$.

7. Calculate the Lagrange equations.

Continuous model Let $q(t,x)$ be a smooth function which verifies $q(x = i\delta, t) = q_i(t)$ for every i.

8. In the limit $\delta \to 0$, show that the Lagrangian can be written

$$L = \int \mathscr{L} \mathrm{d}x \tag{2}$$

where $\mathscr{L}$ is a functional that depends of temporal and spatial derivatives of $q(x,t)$.

9. Contrary to the examples seen during the semester, $\mathscr{L}$ depends of the second spatial derivative of $q(x,t)$. Redo the calculation that gives the Lagrange equations for Lagrangian densities, and generalize it for a density that depends of $\partial_x^2 q$.

10. Show thus

$$\partial_t^2 q(x,t) + \frac{EI}{\mu} \partial_x^4 q(x,t) = 0 \tag{3}$$

11. Give the dispersion relation linking the pulsation ω and the wave vector k for a wave $q(x,t) = q_0 e^{i(\omega t - kx)}$.

12. Boundary conditions impose $\omega^2 \mu \ell^4/(EI) \sim 1$. Deduce the resonant frequency of a steel blade ($E = 2 \times 10^{11}$ Pa, density $\rho = 8 \times 10^3\,\mathrm{kg\,m^{-3}}$) of 10 cm long and 1 mm thick.

13. Explain the noise when a student make its steel ruler oscillate on the edge of the table, while moving the hand holding the end of the ruler toward the inside of the table.

Part 2

Exercise 3: Dynamics of a point on a rotating parabola

We consider the motion of a particle of mass m, moving without friction on a parabola described by the equation $z = x^2/2a$ rotating around the z-axis with angular speed Ω.

14. Give the expression of the Lagrangian of the system as a function of m, g, Ω, x, z, $\dot{x}$ and $\dot{z}$.

15. After eliminating z and $\dot{z}$, give a new expression of this Lagrangian. We shall denote $\lambda = (g/a) - \Omega^2$.

16. Give the Lagrange equation.

17. Put this equation in the form of a first-order system of differential equations by denoting $\dot{x} = y$, and give the expression of dy/dx as a function of λ, x, y and a.

18. Give the expression for the Hamiltonian H of the system, as a function of λ, x, y and a. Is it a constant? Why? Deduce a relation between x^2 and y^2 where a, λ and a constant C are present.

19. Using your answers to the two preceding questions, display the system's trajectories in the phase space for the three cases $\lambda > 0$, $\lambda = 0$ and $\lambda < 0$.

20. In the case where $\lambda = 0$, show that there exists an infinity of equilibrium solutions on the line $y = 0$. Give a physical argument why this can happen (as opposed to the case seen in the lecture where the particle is on a rotating circle).

Exercise 4: Centrifugal governor

This exercise deals with a centrifugal governor (see fig. 2). It consists in 4 articulated arms of negligible mass and length l. The links are assumed to be frictionless. The whole system is fixed at point O and turns around the vertical z-axis with angular speed Ω. This results in moving away the two masses M, and hence in elevating the point A along the vertical axis through a device of mass m. The position of A can be used to control a motor (not displayed on the figure) turning at angular speed Ω and bringing with it the centrifugal governor.

We first consider the device of figure 2 turning at Ω alone, without taking the motor into account.

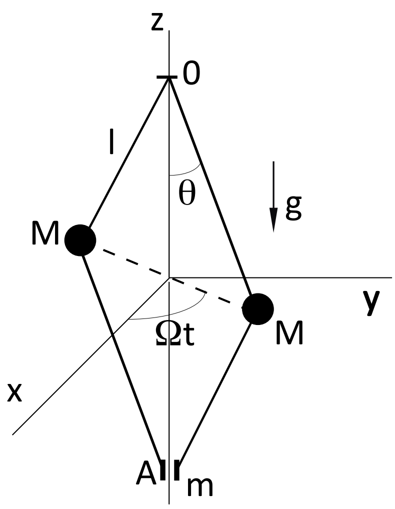

Figure 2: Centrifugal governor.

21. Give the expressions of the coordinates of the masses M and m, and deduce their speeds.

22. Deduce the expression of the Lagrangian of the device in fig. 2.

23. Give the Lagrange equation, and provide a physical interpretation for each term.

24. In particular, discuss the existence of a $\dot{\theta}$ term. Does this correspond to dissipation or amplification? Where could this come from?

We will now consider the effect of the position of A on the angular speed of the motor. The equation modeling this is:

$$J\dot{\Omega} = k\cos\theta - \Gamma, \tag{4}$$

J being (approximately) the moment of inertia of the motor's rotor, $k\cos\theta$ is the motor torque (regulated by the position of A) and Γ the resisting torque (constant friction). We assume $\Gamma < k$. From now on, we set $m = 0$ to simplify the calculations.

25. In addition, we assume that the device from fig. 2 is now submitted to fluid friction with coefficient γ. Show that the system of equation of the problem reads:

$$\dot{\theta} = \phi, \tag{5}$$

$$\dot{\phi} = -\gamma\phi + \Omega^2 \cos\theta \sin\theta - \frac{g}{l}\sin\theta, \tag{6}$$

$$J\dot{\Omega} = k\cos\theta - \Gamma. \tag{7}$$

26. Give the expression of the fixed points (equilibrium solutions) of the system. In particular, show how the equilibrium angular speed $\Omega = \Omega_0$ and the equilibrium angle $\theta = \theta_0$ depend on the constants k, Γ, g, l. Discuss the results.

27. We deal with the linear stability of the equilibrium solutions found in the last question. Let us consider the infinitesimal perturbations $\tilde{\theta}$, $\tilde{\phi}$ and $\tilde{\Omega}$

$$\theta = \theta_0 + \tilde{\theta}, \qquad \phi = 0 + \tilde{\phi}, \qquad \Omega = \Omega_0 + \tilde{\Omega}. \tag{8}$$

28. Inject this substitution in the preceding system of equations, and keep only the perturbative linear terms (neglect higher order terms) in $\tilde{\theta}$, $\tilde{\phi}$ et $\tilde{\Omega}$.

29. Look for solutions in the form

$$\tilde{\theta} = Ae^{st}, \qquad \tilde{\phi} = Be^{st}, \qquad \tilde{\Omega} = Ce^{st}. \tag{9}$$

Show that, in order for the system to have non-zero solutions, the determinant of the system of linear equations obtained must be zero.

30. Show that the resulting equation is a degree three polynom in the variable s, whose roots all have negative real parts, and hence cannot trigger an instability (hint: look at the sign of the polynom's coefficients).

31. An oscillating instability of the system occurs when two roots of the polynom are imaginary and complex-conjugate: $s = \pm i\omega$. Why is this so? The degree 3 polynom can then be written as $(s+\nu)(s^2+\omega^2) = 0$.

32. Show that the system needs to be sufficiently damped (γ big enough) for the equilibrium solution at constant angular speed to be stable. Give the expression of the minimal γ value.