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Final Exam

The two parts will contribute in a equal share in the final mark.
Please answer to part 1 and part 2 on separate sheets.
Note that : $\text{argth}' x = -\frac{1}{x^2-1}$.

1 Taking into account dissipation using the formalism of analytical mechanics

Our aim is to show how dissipation can be taken into account within the formalism of analytical mechanics. We first consider a damped harmonic oscillator

$$m\ddot{x} + 2\lambda\dot{x} + kx = 0. \tag{1}$$

1.1 A variational principle and a Lagrangian for a dissipative system

1. Show that equation (1) can be obtained using the following variational principle

$$\delta\left[\int_{t_1}^{t_2} y(m\ddot{x} + 2\lambda\dot{x} + kx)dt\right] = 0, \tag{2}$$

   where $x(t)$ et $y(t)$ are varied independently and $\delta x = \delta y = 0$ for $t = t_1$ and $t = t_2$.
2. What is the equation governing $y(t)$? How is it obtained from the equation governing $x(t)$ using a simple transformation ?
3. Explain why (the full calculation is not necessary) the equations for $x(t)$ and $y(t)$ can be obtained using the following variational principle :

$$\delta\left[\int_{t_1}^{t_2} L(x, y, \dot{x}, \dot{y})dt\right] = 0, \tag{3}$$

   with $L(x, y, \dot{x}, \dot{y}) = m\dot{x}\dot{y} + \lambda(x\dot{y} - \dot{x}y) - kxy$.
4. Use this Lagrangian to find a conserved quantity.
5. Assume that $x(t)$ is a solution of equation (1). Show that a solution for y is given by $y(t) = x(t)\exp\frac{2\lambda}{m}t$.
6. Use the previous result to show that equation (1) can be obtained from the Lagrangian

$$L(x, \dot{x}, t) = \frac{1}{2}(m\dot{x}^2 - kx^2)\exp\frac{2\lambda}{m}t. \tag{4}$$

   Check this by writing explicitly the corresponding Lagrange equation.

1.2 Dissipation or variable mass ?

1. Give the Hamiltonian H corresponding to the Lagrangian (4).
2. We consider the following initial conditions : $x(0) = a$ and $\dot{x}(0) = -\alpha a$ with $\lambda = m\alpha$. We also define $k = m\omega_0^2$. Give the solution $x(t)$ for $\alpha < \omega_0$.
3. Use the expression of $x(t)$ to compute the mean value $< H >$ of the Hamiltonian averaged on one oscillation period.

4. Do you think that this result for $< H >$ could have been guessed ? Is the Lagrangian (4) in agreement with the usual definition of a Lagrangian ? Is the damped harmonic oscillator described by this Lagrangian ? Discuss these different aspects of the problem.
5. Show that the Lagrangian (4) is the one of a harmonic oscillator with a variable mass $m(t)$ and give the expression of $m(t)$.
6. We consider a pendulum of length l and variable mass $m(t)$ ($m(t)$ being an arbitrary function of time). We note $x(t)$ the angle of the pendulum with the vertical direction. Give the equation of motion. Is it invariant in the transformation $t \to -t$?
7. Is the result obtained for $< H >$ easier to interpret in the context of the system with variable mass ?
8. Show that the governing equation for the system of variable mass can be put in the form $\ddot{x}+\dot{s}\dot{x}+\omega_0^2 x = 0$ when x is small. Write this equation in the form of a system of first order differential equations and give the expression of the Liouville equation. What is the non conservative term ?

1.3 Radiation and dissipation

We now consider the harmonic oscillator displayed in figure 1. k is the spring constant, m is the mass that oscillates along the y axis. This oscillator can generate waves on a string attached to the mass m. The string is held under tension T and ρ is its mass per unit length. Waves propagating along the string are governed by the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \tag{5}$$

with $c^2 = T/\rho$.

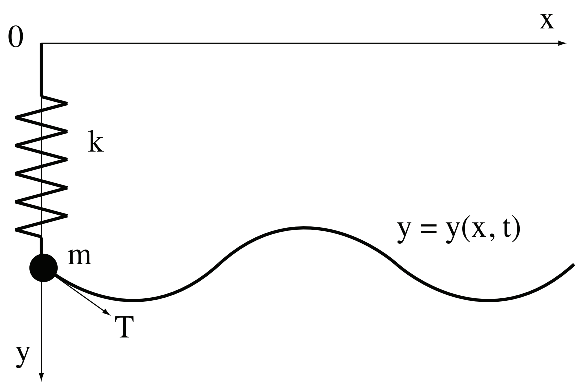


FIGURE 1 – A harmonic oscillator whose mass is attached to a string held under tension T.

1. Give the expression of the vertical component of the force exerted by the string on the oscillator mass m as a function of T and $y(x)$ in the limit of small amplitude oscillations of the string.
2. The general solution of the wave equation is of the form $y(x, t) = f(t-x/c)+g(t+x/c)$. Show that the governing equation for the position $y(0, t)$ of the mass m can be put in the form of an equation for a damped harmonic oscillator with an external forcing.
3. Can you find the expression of the friction coefficient λ (up to a multiplicative constant) without performing the previous calculation ?
4. In which case the external forcing vanishes ? Discuss this result.

2 Inextensible string dynamics : Model of the folding of proteins

We consider a protein, modelled by an inextensible string, free to move in space. As it can bend and twist, we introduce the Frenet frame to follow its movement. The aim of that problem is to determine the energy of the string and to find soliton solutions. Such solitons hint a change in the secondary structure of a protein.

1. **Frenet frame**

   The position on the string is denoted by $\boldsymbol{X}(t,s)$ where s is the arc-length parameter. We define the tangent vector $\boldsymbol{t}(t,s)$ as $\partial_s \boldsymbol{X}(t,s)$ which is unitary ; the normal vector $\boldsymbol{n}(t,s)$ is the unit vector in the direction of $\partial_s \boldsymbol{t}(t,s)$. Finally, the binormal vector $\boldsymbol{b}(t,s)$ by $\boldsymbol{b}(t,s) = \boldsymbol{t}(t,s) \wedge \boldsymbol{n}(t,s)$. The scheme below synthetizes the notations.

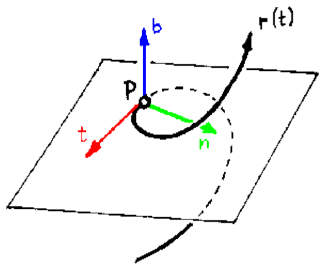


FIGURE 2 – Frenet frame ($\boldsymbol{t}$, $\boldsymbol{n}$, $\boldsymbol{b}$).

   (a) Write $\boldsymbol{t}$, $\boldsymbol{n}$ et $\boldsymbol{b}$ in function of $\boldsymbol{X}$.

   (b) Show that

$$\frac{\partial}{\partial s}\begin{pmatrix}\boldsymbol{t}\\ \boldsymbol{n}\\ \boldsymbol{b}\end{pmatrix} = \begin{pmatrix}0 & \kappa & 0\\ -\kappa & 0 & \tau\\ 0 & -\tau & 0\end{pmatrix}\begin{pmatrix}\boldsymbol{t}\\ \boldsymbol{n}\\ \boldsymbol{b}\end{pmatrix} \tag{6}$$

   with $\kappa(t,s)$ and $\tau(t,s)$ functions called *curvature* and *torsion*. Give their expression.

   (c) Deduce from the previous question that it exists a vector $\boldsymbol{\Omega}$ such that $\partial_s \boldsymbol{t} = \boldsymbol{\Omega} \wedge \boldsymbol{t}$, the same for $\boldsymbol{n}$ and $\boldsymbol{b}$. Explicit this vector. It is called *Darboux' vector*. Ω_1, Ω_2 et Ω_3 correspond respectively to roll, tilt and twist.

2. **Harmonic approximation**

   We search for a simple description which could work for a DNA fragment. Because of its primary structure, the DNA has a natural twist which yields a Darboux' vector at equilibrium $\boldsymbol{\Omega}_0 = (0, 0, \omega)$. Suppose that $\boldsymbol{\Omega}$ is close to $\boldsymbol{\Omega}_0$.

   (a) In this situation, the string accumulates elastic energy. In our case, the equivalent of the stiffness of a spring will be the elastic matrix supposed diagonal of elements A_1, A_2, et A_3. Propose an approximate expression for the energy of the string in which appears the elastic constants and deviations $(\Omega - \Omega_0)_i$ to equilibrium. Emphasize on the justification of this expression.

   (b) The DNA tilt Ω_2 is actually coupled to the twist Ω_3. Propose a modified version of the energy of the string.

3. **Langrangian approach**

   We consider a Lagrangian approach of the problem. We define

$$\psi(t,s) = \kappa(t,s) e^{i \int_0^s \tau(t,s')\,\mathrm{d}s'} \tag{7}$$

   and we consider the Lagrangian density depending on $\psi(s,t)$ and $\overline{\psi}(s,t)$ considered as independent :

$$\mathscr{L} = i\overline{\psi}\partial_t\psi - (\partial_s\psi)(\partial_s\overline{\psi}) + \frac{1}{2}(\psi\overline{\psi})^2 \tag{8}$$

(a) Find the equations of motion.

(b) Give the momenta associated to ψ and $\overline{\psi}$ and deduce the Hamiltonian. Comment.

(c) Which symmetries does this Lagrangian verifies ?

(d) In some theories, we impose that $\int \psi\overline{\psi}\, \mathrm{d}s = 1$. Give the new function to extremize, and the new equations of motions.

4. **Soliton solutions**

Equations of motion can be solved separately for curvature and torsion. In this case the static Lagrangian is :

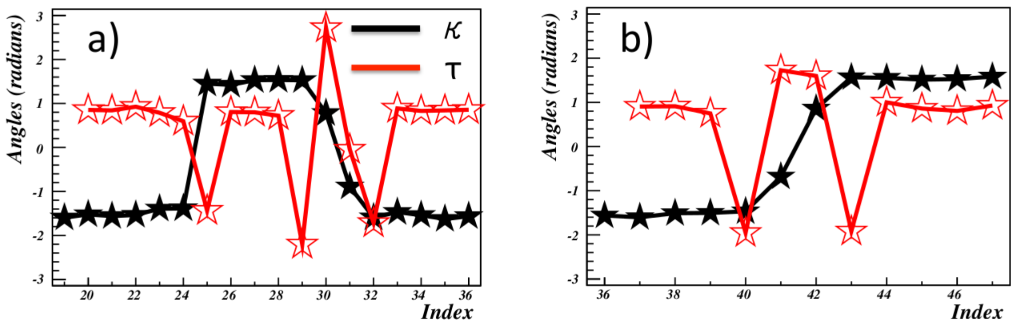
$$L = \int_{-\infty}^{+\infty} \left[(\partial_s \kappa)^2 + \lambda(\kappa^2 - m^2)^2 \right] \mathrm{d}s \tag{9}$$

(a) Find the corresponding equation of motion.

(b) Integrate the equation once, with the boundary condition $\kappa(s) \overset{s\to\infty}{\longrightarrow} m$.

(c) Integrate the equation a second time, to find the soliton expression of the curvature $\kappa(s)$.

(d) The picture below shows the curvature and the torsion of a protein on a series of consecutive sites. Explain the experimental data corresponding to the curvature κ on both graphs, in terms of solitons.



5. **Perturbative development**

The equation studied in the previous paragraphs frequently appears in nonlinear physics. In particular, it can be seen as the equation of propagation of a wave packet in a non-linear medium. Its amplitude A is a function of the position x and the time t. The function A is solution of the following equation :

$$\frac{\partial A}{\partial t} = i\alpha \frac{\partial^2 A}{\partial x^2} - i\beta |A|^2 A \tag{10}$$

where α and β are real constants. There are homogeneous solutions to this equation that can be written $A = Qe^{i\omega t}$ with $\omega = -\beta Q^2$, and we will focus on slightly inhomogeneous solutions, the amplitude of which can be written :

$$A(x,t) = (Q + r(x,t))e^{i(\omega t + \phi(x,t))}. \tag{11}$$

(a) Express the derivatives $\frac{\partial r}{\partial t}$ and $\frac{\partial \phi}{\partial t}$.

We want to do a multiple scale development for r and ϕ. We define the new variables $\xi = \varepsilon(x - ct)$ and $\tau = \varepsilon^3 t$, where $\varepsilon \ll 1$, we define the functions $\tilde{r}(\xi,\tau) = r(x,t)$ with these new variables, and we are looking for a development of $\tilde{r}$ and $\tilde{\phi}$ of the form :

$$\begin{aligned} \tilde{r}(\xi,\tau) &= \varepsilon^2 \left(r_0(\xi,\tau) + \varepsilon^2 r_1(\xi,\tau) + \dots \right) \\ \tilde{\phi}(\xi,\tau) &= \varepsilon \left(\phi_0(\xi,\tau) + \varepsilon^2 \phi_1(\xi,\tau) + \dots \right) \end{aligned} \tag{12}$$

(b) *First order* : Show that :

$$c = Q\sqrt{2\alpha\beta} \qquad \text{et} \qquad r_0 = \sqrt{\frac{\alpha}{2\beta}}\frac{\partial\phi_0}{\partial\xi} \tag{13}$$

(c) *Second order* : Show that r_0 is solution of the following evolution equation :

$$\frac{\partial r_0}{\partial\tau} + \mu r_0\frac{\partial r_0}{\partial\xi} - \nu\frac{\partial^3 r_0}{\partial\xi^3} = 0 \tag{14}$$

where μ and ν are constants to be determined. Which equation is it ?