Another advantage of a mathematical statement is that it is so definite that it might be definitely wrong... Some verbal statements have not this merit.

L. F. Richardson (1881–1953).

CHAPTER 3

Shallow Water Systems and Isentropic Coordinates

CONVENTIONALLY, 'THE' SHALLOW WATER EQUATIONS describe a thin layer of constant density fluid in hydrostatic balance, rotating or not, bounded from below by a rigid surface and from above by a free surface, above which we suppose is another fluid of negligible inertia. Such a configuration can be generalized to multiple layers of immiscible fluids lying one on top of each other, forming a 'stacked shallow water' system, and this class of systems is the main subject of this chapter.

The single-layer model is one of the simplest useful models in geophysical fluid dynamics, because it allows for a consideration of the effects of rotation in a simple framework without with the complicating effects of stratification. By adding layers we can then study the effects of stratification, and indeed the model with just two layers is not only a simple model of a stratified fluid, it is a surprisingly good model of many phenomena in the ocean and atmosphere. Indeed, the models are more than just pedagogical tools — we will find that there is a close physical and mathematical analogy between the shallow water equations and a description of the continuously stratified ocean or atmosphere written in isopycnal or isentropic coordinates, with a meaning beyond a coincidental similarity in the equations. We begin with the single-layer case.

3.1 DYNAMICS OF A SINGLE, SHALLOW LAYER

Shallow water dynamics apply, by definition, to a fluid layer of constant density in which the horizontal scale of the flow is much greater than the depth of the water. The fluid motion is then fully determined by the momentum and mass continuity equations, and because of the assumed small aspect ratio the the hydrostatic approximation is well

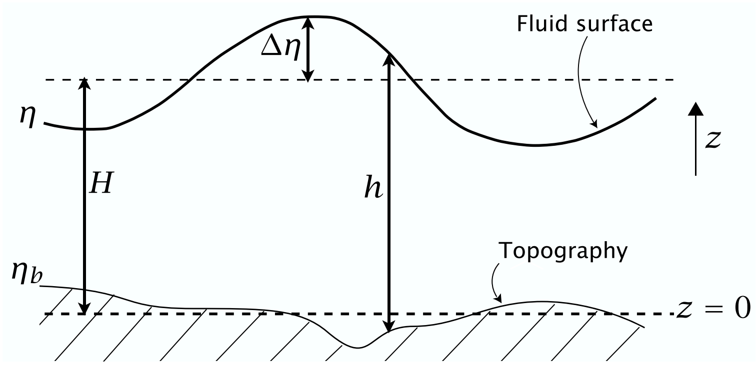

Fig. 3.1 A shallow water system. $h(x, y)$ is the thickness of a water column, H its mean thickness, $\eta(x, y)$ the height of the free surface and η_b is the height of the lower, rigid, surface, above some arbitrary origin, typically chosen such that the average of η_b is zero. $\Delta\eta$ is the deviation free surface height, so we have $\eta = \eta_b + h = H + \Delta\eta$.

satisfied, and we invoke this from the outset. Consider, then, fluid in a container above which is another fluid of negligible density (and therefore negligible inertia) relative to the fluid of interest, as illustrated in Fig. 3.1. As usual, our notation is that $\boldsymbol{v} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$ is the three dimensional velocity and $\boldsymbol{u} = u\mathbf{i} + v\mathbf{j}$ is the horizontal velocity. $h(x, y)$ is thickness of the liquid column, H is its mean height, and η is the height of the free surface. In a flat-bottomed container $\eta = h$, whereas in general $h = \eta - \eta_b$, where η_b is the height of the floor of the container.

3.1.1 Momentum equations

The vertical momentum equation is just the hydrostatic equation,

$$\frac{\partial p}{\partial z} = -\rho g, \tag{3.1}$$

and, because density is assumed constant, we may integrate this to

$$p(x, y, z) = -\rho g z + p_o \tag{3.2}$$

At the top of the fluid, $z = \eta$, the pressure is determined by the weight of the overlying fluid and this is assumed negligible. Thus, $p = 0$ at $z = \eta$ giving

$$p(x, y, z) = \rho g(h(x, y) - z) \tag{3.3}$$

The consequence of this is that the horizontal gradient of pressure is independent of height. That is

$$\nabla_z p = \rho g \nabla_z \eta \tag{3.4}$$

where

$$\nabla_z = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} \tag{3.5}$$

is the gradient operator at constant z. (In the rest of this chapter we will drop the subscript z unless that causes ambiguity. The three-dimensional gradient operator will be denoted ∇_3. We will also mostly use Cartesian coordinates, but the shallow water equations may certainly be applied over a spherical planet — indeed, 'Laplace's tidal equations' are essentially the shallow water equations on a sphere.) The horizontal momentum equations therefore become

$$\frac{D\boldsymbol{u}}{Dt} = -\frac{1}{\rho}\nabla p = -g\nabla\eta \tag{3.6}$$

The right-hand side of this equation is independent of the vertical coordinate z. Thus, if the flow is initially independent of z, it must stay so. (This z-independence is unrelated to that arising from the rapid rotation necessary for the Taylor-Proudman effect.) The velocities u and v are functions only of x, y and t and the horizontal momentum equation is therefore

$$\frac{D\boldsymbol{u}}{Dt} = \frac{\partial \boldsymbol{u}}{\partial t} + u\frac{\partial \boldsymbol{u}}{\partial x} + v\frac{\partial \boldsymbol{u}}{\partial y} = -g\nabla\eta. \tag{3.7}$$

That the horizontal velocity is independent of z is a consequence of the hydrostatic equation, which ensures that the horizontal pressure gradient is independent of height. (Another starting point would be to take this independence of the horizontal motion with height as the *definition* of shallow water flow. In real physical situations such independence does not hold exactly — for example, friction at the bottom may induce a vertical dependence of the flow in a boundary layer.) In the presence of rotation (3.7) easily generalizes to

$$\boxed{\frac{D\boldsymbol{u}}{Dt} + \boldsymbol{f} \times \boldsymbol{u} = -g\nabla\eta}, \tag{3.8}$$

where $\boldsymbol{f} = f\mathbf{k}$. Just as with the primitive equations, f may be constant or may vary with latitude, so that on a spherical planet $f = 2\Omega\sin\vartheta$ and on the β-plane $f = f_0 + \beta y$.

3.1.2 Mass conservation equation

From first principles

The mass contained in a fluid column of height h and cross-sectional area A is given by $\int_A \rho h \, d\boldsymbol{A}$ (Fig. 3.2). If there is a net flux of fluid across the column boundary (by advection) then this must be balanced by a net increase in the mass in A, and therefore a net increase in the height of the water column. The mass convergence into the column is given by

$$F_m = \text{Mass flux in} = -\int_S \rho\boldsymbol{u}\cdot d\boldsymbol{S} \tag{3.9}$$

where S is the area of the vertical boundary of the column. The surface area of the column is comprised of elements of area $h\boldsymbol{n}\delta l$, where δl is a line element circumscribing

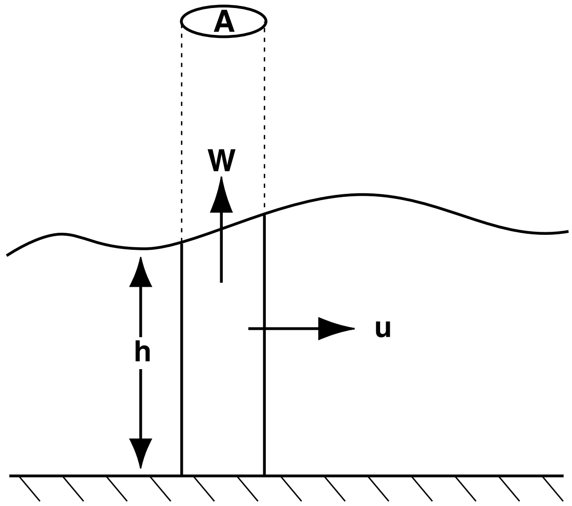

Figure 3.2 The mass budget for a column of area A in a shallow water system. The fluid leaving the column is $\oint \rho h \boldsymbol{u} \cdot \boldsymbol{n} \, \mathrm{d}l$ where $\boldsymbol{n}$ is the unit vector normal to the boundary of the fluid column. There is a non-zero vertical velocity at the top of the column if the mass convergence into the column is non-zero.

the column and $\boldsymbol{n}$ is a unit vector perpendicular to the boundary, pointing outwards. Thus (3.9) becomes

$$F_m = -\oint \rho h \boldsymbol{u} \cdot \boldsymbol{n} \, \mathrm{d}l \tag{3.10}$$

Using the divergence theorem in two-dimensions, (3.10) simplifies to

$$F_m = -\int_A \nabla \cdot (\rho \boldsymbol{u} h) \, \mathrm{d}A. \tag{3.11}$$

where the integral is over the cross-sectional area of the fluid column (looking down from above). This is balanced by the local increase in height of the water column, given by

$$F_m = \frac{\mathrm{d}}{\mathrm{d}t} \int \rho \, \mathrm{d}V = \frac{\mathrm{d}}{\mathrm{d}t} \int_A \rho h \, \mathrm{d}A = \int_A \rho \frac{\partial h}{\partial t} \, \mathrm{d}A \tag{3.12}$$

The balance between (3.11) and (3.12) thus leads to

$$\int_A \left[\frac{\partial}{\partial t} h + \nabla \cdot (\boldsymbol{u} h) \right] \mathrm{d}A = 0 \tag{3.13}$$

Because the area is arbitrary the integrand itself must vanish, whence,

$$\boxed{\frac{\partial h}{\partial t} + \nabla \cdot (\boldsymbol{u} h) = 0}, \tag{3.14}$$

or equivalently

$$\boxed{\frac{\mathrm{D}h}{\mathrm{D}t} + h \nabla \cdot \boldsymbol{u} = 0}. \tag{3.15}$$

This derivation holds whether or not the lower surface is flat. If it is, then $h = \eta$, and if not $h = \eta - \eta_b$. Eqs. (3.8) and (3.14) or (3.15) form a complete set, summarized in the shaded box on page 134.

From the 3D mass conservation equation

Since the fluid is incompressible, the three-dimensional mass continuity equation is just $\nabla \cdot \boldsymbol{v} = 0$. Writing this out in component form

$$\frac{\partial w}{\partial z} = -\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = -\nabla \cdot \boldsymbol{u} \tag{3.16}$$

Integrate this from the bottom of the fluid ($z = \eta_b$) to the top ($z = \eta$), noting that the right-hand side is independent of z, to give

$$w(\eta) - w(\eta_b) = -h\nabla \cdot \boldsymbol{u}. \tag{3.17}$$

At the top the vertical velocity is the matarial derivative of the position of a particular fluid element. But the position of the fluid at the top is just η, and therefore (see Fig. 3.2)

$$w(\eta) = \frac{D\eta}{Dt}. \tag{3.18}$$

At the bottom of the fluid we have similarly

$$w(\eta_b) = \frac{D\eta_b}{Dt}, \tag{3.19}$$

where, absent earthquakes and the like, $\partial\eta_b/\partial t = 0$. Using the last two equations, (3.17) becomes

$$\frac{D}{Dt}(\eta - \eta_b) + h\nabla \cdot \boldsymbol{u} = 0 \tag{3.20}$$

or

$$\frac{Dh}{Dt} + h\nabla \cdot \boldsymbol{u} = 0. \tag{3.21}$$

as in (3.15).

3.1.3 A rigid lid

The case where the *upper* surface is held flat by the imposition of a rigid lid is sometimes of interest. The ocean suggests one such example, for here the bathymetry at the bottom of the ocean provides much larger variations in fluid thickness than do the small variations in the height of the ocean surface. Suppose then the upper surface is at a contant height H then, from (3.14) with $\partial h/\partial t = 0$ the mass conservation equation becomes

$$\nabla_h \cdot (\boldsymbol{u} h_b) = 0. \tag{3.22}$$

where $h_b = H - \eta_b$ Note that this allows us to define an incompressible *mass-transport velocity,* $\boldsymbol{U} \equiv h_b \boldsymbol{u}$.

Although the upper surface is flat, the pressure there is no longer constant because a force must be provided by the rigid lid to keep the surface flat. The horizontal momentum equation is

$$\frac{D\boldsymbol{u}}{Dt} = -\frac{1}{\rho_0}\nabla p_{\text{lid}} \tag{3.23}$$

The Shallow Water Equations

For a single-layer fluid, and including the Coriolis term, the inviscid shallow water equations are:
Momentum:

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + \boldsymbol{f} \times \boldsymbol{u} = -g\nabla\eta. \tag{SW.1}$$

Mass Conservation:

$$\frac{\mathrm{D}h}{\mathrm{D}t} + h\nabla\cdot\boldsymbol{u} = 0 \qquad \text{or} \qquad \frac{\partial h}{\partial t} + \nabla\cdot(h\boldsymbol{u}) = 0. \tag{SW.2}$$

where $\boldsymbol{u}$ is the horizontal velocity, h is the total fluid thickness, η is the height of the upper free surface and η_b is the height of the lower surface (the bottom topography). Thus, $h(x, y, t) = \eta(x, y, t) - \eta_b(x, y)$. The material derivative is

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}, \tag{SW.3}$$

with the rightmost expresion holding in Cartesian coordinates.

where p_{lid} is the pressure at the lid, and the complete equations of motion are then (3.22) and (3.23).[1] If the lower surface is flat, the two-dimensional flow itself is divergence-free, and the equations reduce to the two-dimensional incompressible Euler equations.

3.1.4 Stretching and the vertical velocity

Because the horizontal velocity is depth independent, the vertical velocity plays no role in advection. However, w is certainly not zero for then the free surface would be unable to move up or down, but because of the vertical independence of the horizontal flow w does have a simple vertical structure; to determine this we write the mass conservation equation as

$$\frac{\partial w}{\partial z} = -\nabla\cdot\boldsymbol{u} \tag{3.24}$$

and integrate upwards from the bottom to give

$$w = w_b - (\nabla\cdot\boldsymbol{u})(z - \eta_b). \tag{3.25}$$

Thus, the vertical velocity is a linear function of height. Eq. (3.25) can be written

$$\frac{\mathrm{D}z}{\mathrm{D}t} = \frac{\mathrm{D}\eta_b}{\mathrm{D}t} - (\nabla\cdot\boldsymbol{u})(z - \eta_b), \tag{3.26}$$

and at the upper surface $w = \mathrm{D}\eta/\mathrm{D}t$ so that here we have

$$\frac{\mathrm{D}\eta}{\mathrm{D}t} = \frac{\mathrm{D}\eta_b}{\mathrm{D}t} - (\nabla\cdot\boldsymbol{u})(\eta - \eta_b), \tag{3.27}$$