Arnaud RAOUX (arnaud.raoux@sorbonne-universite.fr)
http://www.phys.ens.fr/~raoux/

Exercise class 3: Berry phase and Foucault Pendulum

1 Magnetic moment in a magnetic field

We study a fixed particle with a spin-1/2. Let $\{|+\rangle, |-\rangle\}$ the eigenvectors of the spin 1/2 operator along the z axis. In this basis, the Pauli matrices yield

$$(\sigma_x) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad (\sigma_y) = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad (\sigma_z) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1}$$

1. If $\boldsymbol{n}$ is a unit vector of spherical coordinates $(1, \theta, \varphi)$, express the matrix $(S_{\boldsymbol{n}})$ that represents the spin operator in the $\boldsymbol{n}$ direction. Find the associated eigenvalues. One can show that the following vectors are eigenvectors of $S_{\boldsymbol{n}}$:

$$\begin{cases} |+_{\boldsymbol{n}}\rangle = e^{-i\varphi/2}\cos\frac{\theta}{2}|+\rangle + e^{i\varphi/2}\sin\frac{\theta}{2}|-\rangle \\ |-_{\boldsymbol{n}}\rangle = -e^{-i\varphi/2}\sin\frac{\theta}{2}|+\rangle + e^{i\varphi/2}\cos\frac{\theta}{2}|-\rangle \end{cases}. \tag{2}$$

> *Correction.*
> The $\boldsymbol{n}$ vector can be decomposed as $\boldsymbol{n} = n_x\boldsymbol{e}_x + n_y\boldsymbol{e}_y + n_z\boldsymbol{e}_z$. As we know the spin operator along the x, y and z coordinates (S_x, S_y and S_z), we deduce that $\hat{S}_{\boldsymbol{n}} = \left(\frac{\hbar}{2}\hat{\boldsymbol{\sigma}}\right)\cdot\boldsymbol{n}$. In terms of matrices:
>
> $$(S_{\boldsymbol{n}}) = (\boldsymbol{\sigma})\cdot\boldsymbol{n} = \frac{\hbar}{2}\begin{pmatrix} \cos\theta & \mathrm{e}^{-i\varphi}\sin\theta \\ \mathrm{e}^{i\varphi}\sin\theta\sin\theta & -\cos\theta \end{pmatrix}$$
>
> The eigenvalues of $(S_{\boldsymbol{n}})$ are $\pm\hbar/2$.

2. We now impose a magnetic field $\boldsymbol{B}(t) = B\boldsymbol{n}(t)$ on the spin, with a direction $\boldsymbol{n}(t)$ that varies in time. Write down the Hamiltonian.

> *Correction.*
> The magnetic moment is proportionnal to the angular moment and can be written: $\hat{\boldsymbol{\mu}} = g\gamma\hat{\boldsymbol{S}}$, where $\gamma = -\frac{e}{2m}$ is the gyromagnetic factor and $g \approx 2$ the Landé factor. Thus:
>
> $$\hat{H} = -\hat{\boldsymbol{\mu}}\cdot\boldsymbol{B} = -g\frac{-eB}{2m}\boldsymbol{S}\cdot\boldsymbol{n} = \frac{\hbar\omega}{2}\boldsymbol{\sigma}\cdot\boldsymbol{n}$$
>
> with $\omega = \frac{eB}{m}$ and $g \approx 2$.

3. The spin is initially in the $|\psi\rangle = |+_{\boldsymbol{n}}\rangle$ state. We change the magnetic field direction smoothly such that the spin *adiabatically* stays in this state, so only its phase can vary. If we write $|\psi(t)\rangle = \alpha(t)|+_{\boldsymbol{n}}\rangle$, give the temporal evolution of $\alpha(t)$.

Correction.
The time evolution is given by the Schrödinger equation: $i\hbar\frac{\mathrm{d}\psi}{\mathrm{d}t} = \hat{H}\psi$. If $|\psi\rangle = \alpha(t)\,|+_{\boldsymbol{n}}\rangle$, then the previous equation becomes

$$i\hbar\dot{\alpha} + i\hbar\alpha\frac{\mathrm{d}\,|+_{\boldsymbol{n}}\rangle}{\mathrm{d}t} = E_+(t)\alpha\,|+_{\boldsymbol{n}}\rangle$$

Taking the scalar product with $|+_{\boldsymbol{n}}\rangle$, this equation becomes:

$$\frac{\dot{\alpha}}{\alpha} = -\frac{i}{\hbar}E_+ - \left\langle +_{\boldsymbol{n}} \middle| \frac{\mathrm{d}}{\mathrm{d}t} \middle| +_{\boldsymbol{n}} \right\rangle = -\frac{i}{\hbar}E_+ - \dot{\boldsymbol{n}}\cdot\langle +_{\boldsymbol{n}}|\nabla_{\boldsymbol{n}}|+_{\boldsymbol{n}}\rangle = -\frac{i}{\hbar}E_+ + \frac{i}{\hbar}\dot{\boldsymbol{n}}\cdot\boldsymbol{\mathcal{A}}_+$$

4. Calculate the Berry connection $\boldsymbol{\mathcal{A}}_+ = i\hbar\,\langle +_{\boldsymbol{n}}|\boldsymbol{\nabla}_{\boldsymbol{n}}|+_{\boldsymbol{n}}\rangle$ where the derivatives refer to the coordinates of the unit vector $\boldsymbol{n}$. If we modify the chosen basis such that $|+_{\boldsymbol{n}}\rangle \to \mathrm{e}^{i\varphi/2}\,|+_{\boldsymbol{n}}\rangle$, how is $\boldsymbol{\mathcal{A}}_+$ modified?

Correction.
We need to write down the derivatives of $|+_{\boldsymbol{n}}\rangle$:

$$\begin{cases} \partial_\theta\,|+_{\boldsymbol{n}}\rangle = -\frac{1}{2}\,\mathrm{e}^{-i\varphi/2}\sin\frac{\theta}{2}\,|+\rangle + \frac{1}{2}\,\mathrm{e}^{i\varphi/2}\cos\frac{\theta}{2}\,|-\rangle \\ \partial_\varphi\,|+_{\boldsymbol{n}}\rangle = -\frac{i}{2}\,\mathrm{e}^{-i\varphi/2}\cos\frac{\theta}{2}\,|+\rangle + \frac{i}{2}\,\mathrm{e}^{i\varphi/2}\sin\frac{\theta}{2}\,|-\rangle \end{cases}$$

Then, it is possible to compute the Berry connection:

$$\boldsymbol{\mathcal{A}}_+ = i\hbar\begin{pmatrix}\langle +_{\boldsymbol{n}}|\partial_\theta|+_{\boldsymbol{n}}\rangle \\ \langle +_{\boldsymbol{n}}|\partial_\varphi|+_{\boldsymbol{n}}\rangle\end{pmatrix} = i\hbar\begin{pmatrix} -\frac{1}{2}\sin\frac{\theta}{2}\cos\frac{\theta}{2} + \frac{1}{2}\sin\frac{\theta}{2}\cos\frac{\theta}{2} \\ -\frac{i}{2}\left(\cos\frac{\theta}{2}\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\sin\frac{\theta}{2}\right)\end{pmatrix} = \hbar\cos\theta\nabla\varphi = \hbar\frac{\cos\theta}{r\sin\theta}\boldsymbol{e}_\varphi$$

If we modify the basis, then the connection is also modified. In the proposed case: $\boldsymbol{\mathcal{A}}_+ = \frac{\hbar}{2}(\cos\theta - 1)\nabla\varphi$. This last expression reminds us of the vector potential of a magnetic monopole (with a Dirac string $z < 0$)…
We can establish an analogy between the magnetic vector potential of a monopole in real space, and the Berry connection in the parameter space (parameter space being the coordinates of $\boldsymbol{n}$).

5. Calculate the Berry curvature $\boldsymbol{\mathcal{B}}_+ = \boldsymbol{\nabla}\times\boldsymbol{\mathcal{A}}_+$. Does it depend on the choice of φ?

Correction.

$$\boldsymbol{\mathcal{B}}_+ = \nabla\times\boldsymbol{\mathcal{A}}_+ = \nabla\times\frac{\hbar}{2}\frac{\cos\theta}{r\sin\theta}\boldsymbol{e}_\varphi = \frac{\hbar}{2}\frac{\boldsymbol{e}_r}{r^2}$$

This is again a similar expression to the $\boldsymbol{B}$ field of a magnetic monopole!
$\boldsymbol{\mathcal{B}}_+$ is a curvature, and is gauge-independent. It does not depend on the choice of the basis.

6. Suppose now that the $\boldsymbol{n}(t)$ vector describes a closed path on the unit sphere. Calculate the accumulated phase of the spin. Justify that one decomposes the result in *geometrical* and *dynamical* contributions.

Correction.
Let's come back to the equation found in question **3**, and integrate:

$$\frac{\alpha(t)}{\alpha_0} = \exp\frac{i}{\hbar}\left(-\int E_+(t)\mathrm{d}t + \int \dot{\boldsymbol{n}}\cdot\boldsymbol{\mathcal{A}}_+\mathrm{d}t\right) = \underbrace{\exp\left(-\frac{i}{\hbar}\int E_+(t)\mathrm{d}t\right)}_{=i\chi_{\mathrm{dyn}}}\underbrace{\exp\left(\frac{i}{\hbar}\oint_\gamma \boldsymbol{\mathcal{A}}_+\cdot\mathrm{d}\boldsymbol{\ell}\right)}_{=i\chi_{\mathrm{geo}}}$$

with χ_{dyn} a phase that actually depends on the dynamics of the magnetic field and the velocity at which the direction changes, and χ_{geo} a phase in which t has disappeared, and only depends on the trajectory followed by the vector $\boldsymbol{n}(t)$, whatever its velocity. This last contribution is thus *geometrical*.

7. What is the condition for the adiabatic approximation to be verified?

Correction.
For the adiabatic condition to be verified, one needs to have a slow dynamics of the magnetic field.
To be more concrete, one needs to introduce the full Berry connection tensor. If we define: $\boldsymbol{\mathcal{A}}_{+-} = i\hbar\langle +_{\boldsymbol{n}}|\boldsymbol{\nabla}_{\boldsymbol{n}}|-_{\boldsymbol{n}}\rangle$, we need the rate of change $|\dot{\boldsymbol{n}}\cdot\boldsymbol{\mathcal{A}}_{+-}|$ to be small compared to the Bohr pulsation associated to the gap between the two states: $\frac{1}{\hbar}(E_+ - E_-) = \omega$. Thus, the time scale of the dynamics of the magnetic field should be much longer than the time scale associated with the gap.
Sorry for my confusion during the class. It came from the wrong idea that ω is the quantity driving the transition between the two states, whereas it is obviously the other quantity. The bigger the rate of change of the field, the higher the energy for the spin, and if it equals the gap, then the system can jump on the other state.

2 Foucault Pendulum: an example of parallel transport

The Foucault pendulum is a simple device conceived in 1851 as an experiment to demonstrate the Earth's rotation. Foucault suspended a 28-kilogram bob with a 67-metre long wire from the dome of the Panthéon, Paris. Because the latitude of its location, the plane of the pendulum's swing rotates clockwise at approximately 11,3° per hour.

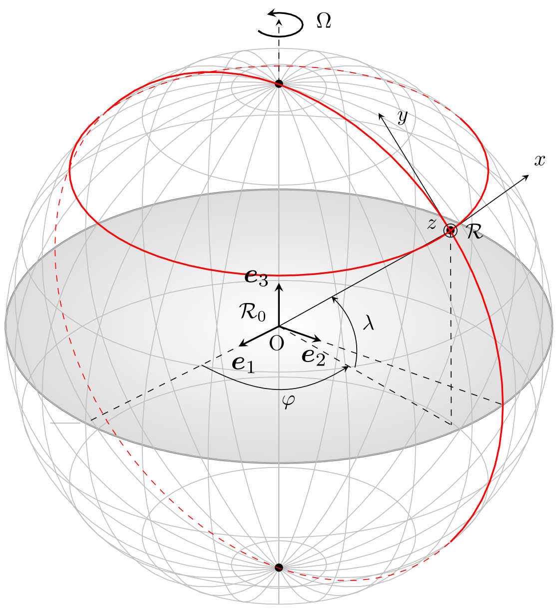

Figure 1: The Earth is rotating at Ω around the $\boldsymbol{e}_3$ axis. $\mathcal{R}_0$ and $\mathcal{R}$ rotate with the sphere.

Classical treatment

We follow the position of a point on a sphere at latitude λ with the local base $(\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z)$, $\boldsymbol{e}_z$ being in the vertical direction. Let $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ be a fixed base in the frame of reference $\mathcal{R}_0$ in which the center of gravity of the Earth is at rest (*le référentiel géocentrique*) and such that the sphere is rotating at an angular speed Ω around $\boldsymbol{e}_3$ in $\mathcal{R}_0$. We denote $\mathcal{R}$ the frame that follows the local base $(\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z)$ (*le référentiel terrestre*).

8. Within the approximation of a planar and harmonic motion, write the equations of motion of the system. Show that they can be assembled to give a single equation in terms of $u(t) = x + iy$:

$$\ddot{u} + 2i\,\dot{u}\,\tilde{\omega} + \omega^2 u = 0. \tag{3}$$

Correction.
The forces that act on the system are its weight $\boldsymbol{P}$ and the tension of the string $\boldsymbol{T}$. In the frame $\mathcal{R}$, we need to add inertial contributions as the Coriolis force to Newton's second law (the centrifugal force is already taken into account in the weight). When summed, the two first ones act at first approximation as a planar restoring force of a harmonic oscillator. Thus, the equation of motion in the plane of the motion can be written

$$m\left.\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}\right|_{\mathcal{R}} = -m\omega^2\boldsymbol{r} - 2m\boldsymbol{\Omega}\times\boldsymbol{v}$$

with $\boldsymbol{e}_3 = \sin\theta\boldsymbol{e}_y + \cos\theta\boldsymbol{e}_z$, and $\boldsymbol{r}$ the position vector of polar coordinates on the tangent plane.
Assuming that the motion is planar: $m\left.\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}\right|_{\mathcal{R}}$ should be in the tangent plane, so we approximate $\boldsymbol{e}_3\times\boldsymbol{v} \approx \cos\theta\boldsymbol{e}_z\times\boldsymbol{v} = \cos\theta(\dot{x}\boldsymbol{e}_y - \dot{y}\boldsymbol{e}_x)$. The equations of motion are thus

$$\begin{aligned}\ddot{x} &= -\omega^2 x + 2\Omega\cos\theta\dot{y}\\ \ddot{y} &= -\omega^2 y - 2\Omega\cos\theta\dot{x}\end{aligned}$$

Finally, by defining $u = x + iy$, one finds the equation needed.
Note that if at first sight u is only an intermediate quantity to simplify the calculation, it actually has a strong meaning. u is the complex coordinate in the plane xOy, thus the phase of u give the orientation of the plane of oscillations of the pendulum!

9. Solve this equation, and express the direction of the plane of oscillations in function of time. Calculate the rotation of the plane of the pendulum in Paris (latitude 48°), after one day of oscillations.

Correction.
One can solve this equation exactly, but it is maybe more physical to find an approximate solution identifying two different time scales. The first one is the period of oscillation of the pendulum ($T_1 = 2\pi/\omega$), and the second one the rotation period of Earth ($T_2 = 2\pi/\Omega$). Thus, we can suggest an ansatz for $u(t) = v(t)\,\mathrm{e}^{i\omega t}$, $v(t)$ evolving at time scale T_2. Then the equation becomes:

$$\ddot{v} + 2i(\Omega + \omega)\dot{v} - 2\tilde{\Omega}\omega v = 0$$

Now, using the fact that $\ddot{v}$ evolves at Ω (and $\omega \gg \Omega$), the previous equation simplifies in $\dot{v} = -i\tilde{\Omega}v$, which gives:

$$u(t) = v(0)\,\mathrm{e}^{i\omega t}\,\mathrm{e}^{-i\tilde{\Omega}t} = v(0)\,\mathrm{e}^{i\omega t}\,\mathrm{e}^{-i\Omega\sin\lambda t}$$

where we replaced the co-latitude angle θ by the latitude $\lambda = \pi/2 - \theta$. So the plane of oscillations is rotating with the planet, but at a different angular velocity, and in the retrograd direction!
After one day ($t = 2\pi/\Omega$), for Paris latitude ($\lambda = 48°$), we get an angle $-2\pi\sin\theta \approx -267°$.

Parallel transport

This classical result can actually be understood as a holonomy problem on a curved surface (a sphere). We don't consider the dynamical oscillations of the pendulum here. Let γ be a circle on a sphere at fixed latitude λ, $x = \gamma(s)$ a point on the circle and $T_x\mathcal{S}^2$ the tangent plane at x.

Let's remember that the frame $\mathcal{R}_0$ is fixed, but $\mathcal{R}$ follows $\mathcal{T}_x\mathcal{S}^2$. The derivatives of a vector $\boldsymbol{U}$ in the different frames of reference are linked by the equation:

$$\left.\frac{\mathrm{d}\boldsymbol{U}}{\mathrm{d}s}\right|_{\mathcal{R}} = \left.\frac{\mathrm{d}\boldsymbol{U}}{\mathrm{d}s}\right|_{\mathcal{R}_0} + \boldsymbol{U} \times \boldsymbol{\Omega} \tag{4}$$

10. Let $\boldsymbol{V}(s)$ be the direction of the plane of oscillations of the pendulum at x. If P_x is the projector on $\mathcal{T}_x\mathcal{S}^2$ at x, justify that

$$P_x \left.\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}s}\right|_{\mathcal{R}_0} = 0 \tag{5}$$

defines a unique way to connect different tangent spaces one to another. This connection is called the *Levi-Civita connection*.

> *Correction.*
> In the $\mathcal{R}_0$ frame, Earth is moving, so $\mathcal{T}_x$ and $\mathcal{T}_{x+\mathrm{d}x}$ are not two identical planes.
> A connection is a way to linked two vectors that don't live on the same space. Let's make a copy of $\mathcal{T}_{x+\mathrm{d}x}$ that intersects with $\mathcal{T}_x$ such that the origin of $\boldsymbol{V}(x)$ and $\boldsymbol{V}(x+\mathrm{d}x)$ are superposed. Defining the connection means defining the vector $\boldsymbol{V}(x+\mathrm{d}x)$. If we impose that $\boldsymbol{V}(x+\mathrm{d}x)$ is the orthogonal projection of $\boldsymbol{V}(x)$ on $\mathcal{T}_{x+\mathrm{d}x}$, $\boldsymbol{V}(x+\mathrm{d}x)$ is uniquely defined.

11. Simplify $P_x \left.\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}s}\right|_{\mathcal{R}}$. Show that

$$\left.\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}s}\right|_{\mathcal{R}} = \tilde{\Omega}\boldsymbol{V} \times \boldsymbol{n}_x \tag{6}$$

where $\boldsymbol{n}_x$ is the local normal vector to $\mathcal{T}_x\mathcal{S}^2$, and $\tilde{\Omega}$ is a constant to determine. Describe the evolution of the vector $\boldsymbol{V}$ along γ.

> *Correction.*
> In the frame $\mathcal{R}$, the motion of x is always in the tangent space (assuming that the motion is planar). Then $\boldsymbol{V}(x)$ and $\boldsymbol{V}(x+\mathrm{d}x)$ are in the same tangent space, as well as the derivative. Thus: $P_x \left.\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}s}\right|_{\mathcal{R}} = \left.\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}s}\right|_{\mathcal{R}}$.
> Let's now apply the projector $\mathcal{P}_x$ on Eq. (4). With the two previous results:
>
> $$\left.\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}s}\right|_{\mathcal{R}} = \mathcal{P}_x(\boldsymbol{V} \times \boldsymbol{\Omega}) = \Omega \sin\lambda \boldsymbol{V} \times \boldsymbol{n})$$
>
> where we used in the last equality that the projection on $\mathcal{T}_x$ is obtained by keeping only the projection of $\boldsymbol{\Omega}$ on $\boldsymbol{e}_z$.
> This equation is caracteristic of a rotating vector at angular velocity $\Omega \sin\lambda$.

12. The holonomy of a closed path γ on the unit sphere is given by: $h(\gamma) = \iint_S \mathrm{d}\Omega$ with S the surface described by γ and $\mathrm{d}\Omega$ the solid angle. Calculate the holonomy of the path γ on the sphere passing through Paris. Comment.

> *Correction.*
> The holonomy of a circle at fixed λ is the area of the top cap of the sphere:
>
> $$h(\gamma) = \iint_S \mathrm{d}\Omega = \int_0^{\pi/2-\lambda} 2\pi \sin\theta \mathrm{d}\theta = 2\pi(1 - \sin(\theta) = 2\pi - 2\pi\sin\lambda$$
>
> There are two parts in this result: 2π that comes from the Earth's rotation itself, and $-2\pi\sin\lambda$ the retrograd contribution showing the anholonomy on a sphere (Anholonomy: for a closed path γ such that $\gamma(a) = \gamma(b)$, a vector field $\boldsymbol{V}$ verifies $\boldsymbol{V}(a) \neq \boldsymbol{V}(b)$).