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Exercise class 3: Berry phase and Foucault Pendulum

1 Magnetic moment in a magnetic field

We study a fixed particle with a spin-1/2. Let $\{|+\rangle, |-\rangle\}$ the eigenvectors of the spin 1/2 operator along the z axis. In this basis, the Pauli matrices yield

$$(\sigma_x) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad (\sigma_y) = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad (\sigma_z) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1}$$

1. If $\boldsymbol{n}$ is a unit vector of spherical coordinates $(1, \theta, \varphi)$, express the matrix $(S_{\boldsymbol{n}})$ that represents the spin operator in the $\boldsymbol{n}$ direction. Find the associated eigenvalues. One can show that the following vectors are eigenvectors of $S_{\boldsymbol{n}}$:

$$\begin{cases} |+_{\boldsymbol{n}}\rangle = e^{-i\varphi/2} \cos\frac{\theta}{2} |+\rangle + e^{i\varphi/2} \sin\frac{\theta}{2} |-\rangle \\ |-_{\boldsymbol{n}}\rangle = -e^{-i\varphi/2} \sin\frac{\theta}{2} |+\rangle + e^{i\varphi/2} \cos\frac{\theta}{2} |-\rangle \end{cases} . \tag{2}$$

2. We now impose a magnetic field $\boldsymbol{B}(t) = B\boldsymbol{n}(t)$ on the spin, with a direction $\boldsymbol{n}(t)$ that varies in time. Write down the Hamiltonian.

3. The spin is initially in the $|\psi\rangle = |+_{\boldsymbol{n}}\rangle$ state. We change the magnetic field direction smoothly such that the spin *adiabatically* stays in this state, so only its phase can vary. If we write $|\psi(t)\rangle = \alpha(t) |+_{\boldsymbol{n}}\rangle$, give the temporal evolution of $\alpha(t)$.

4. Calculate the Berry connection $\boldsymbol{\mathcal{A}}_+ = i\hbar \langle +_{\boldsymbol{n}} | \boldsymbol{\nabla}_{\boldsymbol{n}} | +_{\boldsymbol{n}} \rangle$ where the derivatives refer to the coordinates of the unit vector $\boldsymbol{n}$. If we modify the chosen basis such that $|+_{\boldsymbol{n}}\rangle \to \mathrm{e}^{i\varphi/2} |+_{\boldsymbol{n}}\rangle$, how is $\boldsymbol{\mathcal{A}}_+$ modified?

5. Calculate the Berry curvature $\boldsymbol{\mathcal{B}}_+ = \boldsymbol{\nabla} \times \boldsymbol{\mathcal{A}}_+$. Does it depend on the choice of φ?

6. Suppose now that the $\boldsymbol{n}(t)$ vector describes a closed path ont he unit sphere. Calculate the accumulated phase of the spin. Justify that one decomposes the result in *geometrical* and *dynamical* contributions.

7. What is the condition for the adiabatic approximation to be verified?

2 Foucault Pendulum: an example of parallel transport

The Foucault pendulum is a simple device conceived in 1851 as an experiment to demonstrate the Earth's rotation. Foucault suspended a 28-kilogram bob with a 67-metre long wire from the dome of the Panthéon, Paris. Because the latitude of its location, the plane of the pendulum's swing rotates clockwise at approximately 11,3° per hour.

Classical treatment

We follow the position of a point on a sphere at latitude λ with the local base $(\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z)$, $\boldsymbol{e}_z$ being in the vertical direction. Let $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ be a fixed base in the frame of reference $\mathcal{R}_0$ in which the center of gravity of the Earth is at rest (*le référentiel géocentrique*) and such that the sphere is rotating at an angular speed Ω around $\boldsymbol{e}_3$.

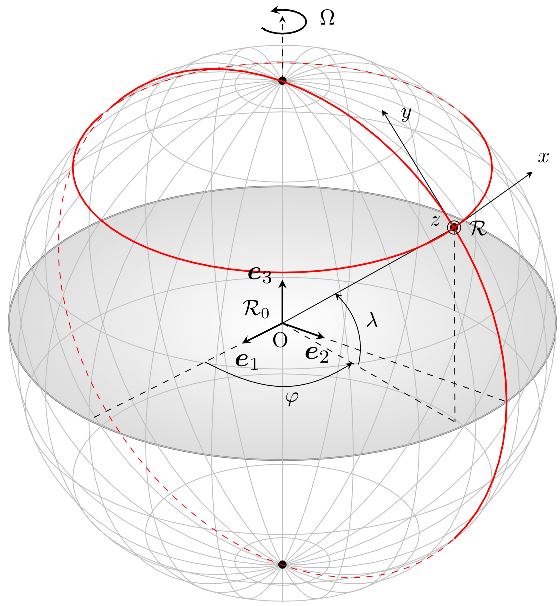

Figure 1: The Earth is rotating at Ω around the $\boldsymbol{e}_3$ axis. $\mathcal{R}_0$ and $\mathcal{R}$ rotate with the sphere.

8. Give the expression of the local vectors in terms of $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$.

9. Within the approximation of a planar and harmonic motion, write the equations of motion of the system. Show that they can be assembled to give a single equation in terms of $u(t) = x + iy$:

$$\ddot{u} + 2i\,\dot{u}\,\tilde{\omega} + \omega^2 u = 0 \tag{3}$$

10. Solve this equation, and express the direction of the plane of oscillations in function of time. Calculate the rotation of the plane of the pendulum in Paris (latitude 48°), after one day of oscillations.

Parallel transport

This classical result can actually be understood as a holonomy problem on a curved surface (a sphere). Let γ be a circle on a sphere at fixed latitude λ. We don't consider the dynamical oscillations of the pendulum here. We denote $x = \gamma(s)$ a point on the circle and $T_x\mathcal{S}^2$ the tangent plane at x.

The frame of reference $\mathcal{R}_0$ is defined as before, and we also define a frame of reference $\mathcal{R}$ that follows $\mathcal{T}_x\mathcal{S}^2$ (*le référentiel terrestre*). The derivatives of a vector $\boldsymbol{U}$ in the different frames of reference are linked by the equation:

$$\left.\frac{\mathrm{d}\boldsymbol{U}}{\mathrm{d}s}\right|_{\mathcal{R}} = \left.\frac{\mathrm{d}\boldsymbol{U}}{\mathrm{d}s}\right|_{\mathcal{R}_0} + \boldsymbol{U} \times \boldsymbol{\Omega} \tag{4}$$

11. Let $\boldsymbol{V}(s)$ be the direction of the plane of oscillations of the pendulum at x. If P_x is the projector on $\mathcal{T}_x\mathcal{S}^2$ at x, justify that

$$P_x \left.\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}s}\right|_{\mathcal{R}_0} = 0 \tag{5}$$

defines a unique way to connect different tangent spaces one to another. This connection is called the *Levi-Civita connection.*

12. Simplify $P_{\gamma(s)} \left.\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}s}\right|_{\mathcal{R}}$. Show that

$$\left.\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}s}\right|_{\mathcal{R}} = \tilde{\Omega}\boldsymbol{V} \times \boldsymbol{n}_x \tag{6}$$

where $\boldsymbol{n}_x$ is the local normal vector to $\mathcal{T}_x\mathcal{S}^2$, and $\tilde{\Omega}$ is a constant to determine. Describe the evolution of the vector $\boldsymbol{V}$ along γ.

13. The holonomy of a closed path γ on the unit sphere using the Levi-Civita connection is given by: $h(\gamma) = \iint_S \mathrm{d}\Omega$ with S the surface described by γ and $\mathrm{d}\Omega$ the solid angle. Calculate the holonomy of the path γ on the sphere passing through Paris. Comment.