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Exercice class 1: Aharonov-Bohm effect

1 Magnetism and analytical mechanics

1. Write the electric and magnetic fields ($\boldsymbol{E}$ and $\boldsymbol{B}$) in terms of the potentials V and $\boldsymbol{A}$. Show that multiple choices of potentials yield the same result for the electromagnetic fields.

> *Correction.*
> Using Maxwell-Thomson equation: $\nabla \cdot \boldsymbol{B}(\boldsymbol{r}) = \boldsymbol{0}$, we know that it exists a vector potential $\boldsymbol{A}$ in the neighborhood of $\boldsymbol{r}$ such that $\boldsymbol{B} = \nabla \times \boldsymbol{A}$. Now, using Maxwell-Faraday equation $\nabla \times \boldsymbol{E} = -\partial_t \boldsymbol{B}$, we conclude that it exists V in the neighborhood of $\boldsymbol{r}$ such that
>
> $$\boldsymbol{B} = \nabla \times \boldsymbol{A} \qquad \text{and} \qquad \boldsymbol{E} = -\nabla V - \partial_t \boldsymbol{A}$$
>
> One can choose any $\boldsymbol{A}'$ that differs from $\boldsymbol{A}$ by a gradient: $\boldsymbol{A}' = \boldsymbol{A} + \nabla f$. In effect, the first equation is still verified. To verify the second one, we need to change at the same time the scalar potential $V' = V - \partial_t f$.
>
> $$\textbf{Gauge freedom:} \quad \begin{cases} \boldsymbol{A}' = \boldsymbol{A} + \nabla f \\ V' = V - \partial_t f \end{cases}$$

2. Give the canonical momentum $\boldsymbol{p}$ and the classical Hamiltonian of a classical charge q in a EM field. One can show that the quantum Hamiltonian has the same expression, changing $\boldsymbol{r}$ and $\boldsymbol{p}$ by the position $\hat{\boldsymbol{r}}$ and momentum $\hat{\boldsymbol{P}}$ operators. Let $\psi(\boldsymbol{r})$ be a solution of the Schrödinger equation. If the potential $\boldsymbol{A}$ undergoes a gauge transformation $\boldsymbol{A} \to \boldsymbol{A} + \boldsymbol{\nabla}\chi$, find the wave-function ψ' that verifies the new equation. Does this change affect the physical quantities of the problem?

> *Correction.*
> By definition: $\boldsymbol{p} = \partial_{\dot{\boldsymbol{r}}} \mathcal{L}$. In the case of $\mathcal{L} = \frac{1}{2} m \dot{\boldsymbol{r}}^2 + q(\dot{\boldsymbol{r}} \cdot \boldsymbol{A}(\boldsymbol{r}) - V(\boldsymbol{r}))$, we get: $\boldsymbol{p} = m\dot{\boldsymbol{r}} + q\boldsymbol{A}$. Thus, the canonical momentum is not the same as the momentum (*quantité de mouvement*). Then, we get the expression of the Hamiltonian:
>
> $$\mathscr{H}(\boldsymbol{r}, \boldsymbol{p}) = \frac{(\boldsymbol{p} - q\boldsymbol{A}(\boldsymbol{r}))^2}{2m} + qV(\boldsymbol{r}).$$
>
> This classical Hamiltonian becomes in quantum mechanics:
>
> $$\hat{H}(\hat{\boldsymbol{r}}, \hat{\boldsymbol{P}}) = \frac{\left(\hat{\boldsymbol{P}} - q\boldsymbol{A}(\hat{\boldsymbol{r}})\right)^2}{2m} + qV(\hat{\boldsymbol{r}}).$$
>
> By a gauge transformation, $\hat{H} \to \hat{H}$. To find the modified wavefunction, we write $\hat{H}\psi = \hat{H}'\psi'$. In particular, let's look at the action of the operator $\hat{\boldsymbol{P}} - q\boldsymbol{A}'(\hat{\boldsymbol{r}})$:
>
> $$\left(\hat{\boldsymbol{P}} - q\boldsymbol{A} - q\nabla f\right)\psi' = (-i\hbar\nabla - q\boldsymbol{A} - q\nabla f)\,\psi' = (-i\hbar\nabla - q\boldsymbol{A})\,\psi$$
>
> if we choose: $\boxed{\psi' = \mathrm{e}^{i\frac{q}{\hbar}\nabla f}\,\psi}$. A global phase let the Physics unchanged!

3. Write the spatio-temporal equation satisfied by the wave-function $\psi(\boldsymbol{r}, t)$. Show that one can define a new derivative operator Δ_μ, $0 \leq \mu \leq 3$ such that the wave-function verifies the Schrödinger equation *without* EM field if ∂_μ is replaced by Δ_μ.

Correction.
The time-dependent Schrodinger equation writes:

$$i\hbar\partial_t\psi = \frac{(-i\hbar\nabla - q\boldsymbol{A})^2}{2m}\psi + qV\psi.$$

If we define:

$$\begin{cases} \nabla \to \nabla'_i = \partial_i - i\frac{q}{\hbar}A_i \\ \partial_t \to \nabla_0 = \partial_t - i\frac{q}{\hbar}(-V) \end{cases},$$

then this definition erases the presence of the electromagnetic field on the wavefunction. Its action is transformed into a change in the connexion linking wavefunctions in the evolution space. $\nabla'_\mu = \partial_\mu - \omega$ is called a **covariant derivative**, with $\omega = i\frac{q}{\hbar}(V, \boldsymbol{A})$ the choosen **connexion**.

2 Aharonov-Bohm effect

2.1 Theoretical description

We consider a situation where a beam of electrons moving in the xy plane enters a region with a solenoid along the z axis. An electron can pass on the left or the right of the solenoid, and then the electronic wave-function is probed. The situation is described in Fig. 1.

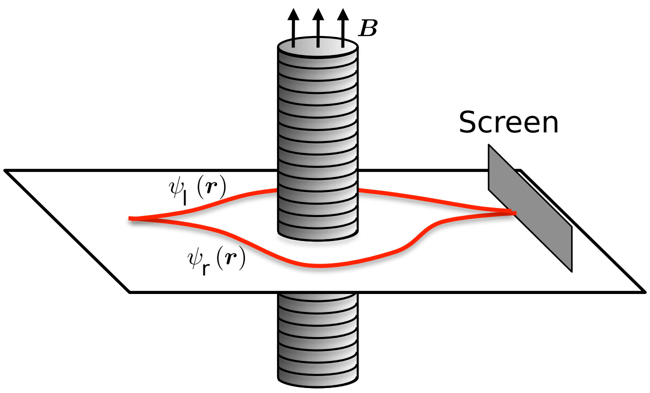

Figure 1: Geometry of the Aharonov-Bohm effect. Adapted from [1].

4. Suppose that the magnetic field is turned off in this question. What is the pattern observed on the screen?

Correction.
Because there is an obstacle, (at least) two different paths are defined for the electrons, which allow to see interferences. Indeed, on the screen one can see an interferogram (fringes).

5. What is the expression of the magnetic field inside and outside the solenoid? Deduce an expression for the vector potential.

Correction.
$\boldsymbol{B} = B_0 \boldsymbol{e}_z = \mu_0 n I \boldsymbol{e}_z$ inside the solenoid, and $\boldsymbol{B} = \boldsymbol{0}$ outside. The only region accessible by the electron is outside the solenoid, so in the accessible space, $\boldsymbol{B} = \boldsymbol{0}$.
A possible solution for the vector potential outside is $\boldsymbol{A} = B_0 \frac{R^2}{2r} \boldsymbol{e}_\varphi$. It is defined everywhere outside the solenoid, and verifies $\nabla \times \boldsymbol{A} = \boldsymbol{B}$. It can not be gauge-away, because it is not a gradient of a function. In fact, writing: $\boldsymbol{A} = \nabla \chi$ gives $\chi = \frac{B_0 R^2}{2} \varphi$. This expression is not a singled-valued function, and so is not acceptable.
As it can not be gauge-away, the solution of the Schrodinger equation is not trivial.

6. In a simply connected space (without holes), every field $\boldsymbol{C}$ verifying $\boldsymbol{\nabla} \times \boldsymbol{C} = \boldsymbol{0}$ can be written in terms of a gradient of an arbitrary function f such as $\boldsymbol{C} = \boldsymbol{\nabla} f$. How can it be applied to our situation?

Correction.
Contrary to the whole AB space, in restricted regions of space that are simply connected (without holes), we can solve the equation: $\nabla \times \boldsymbol{A} = \boldsymbol{0}$ by $\boldsymbol{A} = \nabla \chi$.
Let's define two regions of space: one that goes around the solenoid from the left side, and one from the right side. As any of the regions go around the solenoid, they are simply connected. We can write:

$$\boldsymbol{A}_I = \nabla \chi_I \qquad \text{and} \qquad \boldsymbol{A}_{II} = \nabla \chi_{II}$$

7. Deduce from the previous question expressions for the electronic wave-functions passing to the left ψ_ℓ and to the right ψ_r of the solenoid in the presence of the constant magnetic field in terms of the wave-functions without magnetic field $\psi^{(0)}$.

Correction.
In that case, the vector potentials are pure gradient, and can be gauge-away. So to solve the Schrodinger equation with magnetic field in each region, is equivalent to solve the equation without the magnetic field for a modified wavefunction: $\psi_{l/r} = \mathrm{e}^{i \frac{q}{\hbar} \chi_{I/II}} \, \psi_{l/r}^{(0)}$.

8. Show that the product $\psi_\mathrm{r}^\star \psi_\ell$ does not depend on the choice of the gauge.

Correction.
Because $\boldsymbol{A}_I = \nabla \chi_I$ can be written $\chi_I(\boldsymbol{r}) = \int_0^r \boldsymbol{A}_I \cdot \mathrm{d}\ell$, the product $\psi_\mathrm{r}^\star \psi_\ell$ for particles starting in 0 and arriving at the same point on the screen writes:

$$\psi_\mathrm{r}^\star \psi_\ell = \exp i \frac{q}{\hbar} \underbrace{\left(\int_0^r \boldsymbol{A}_I \cdot \mathrm{d}\ell - \int_0^r \boldsymbol{A}_{II} \cdot \mathrm{d}\ell \right)}_{= \oint \boldsymbol{A} \cdot \mathrm{d}\ell = \iint \boldsymbol{B} \cdot \mathrm{d}S} \psi_\mathrm{r}^{(0)\star} \psi_\ell^{(0)} = \exp \left(i \frac{q}{\hbar} \Phi \right) \psi_\mathrm{r}^{(0)\star} \psi_\ell^{(0)}$$

The prefactor is **gauge-independent**, as it only depends on the magnetic flux. We started with two gauge-dependent quantities that are not observable, to end with a gauge-independent one, that can be seen experimentally!
As this term is responsible for the interference pattern on the screen, the prefactor will shift the pattern without magnetic field.

2.2 Study of Tonomura et al.

9. Identify the structure of the article, and separate the parts: Introduction, Methods, Results, Discussion, Conclusion.

10. How is the image on Fig. 4 obtained?

> *Correction.*
> A collimated electron beam passes near a toroidal magnet. Some part of the wavefunction passes inside the magnet. They use a biprism to create interferences between the two beams.

11. What does Fig. 1(b) show?

> *Correction.*
> The magnetic lines of the magnet. The authors present this result to show that there is no magnetic field-leakage outside of the magnet.

12. What is the main result of the article presented on Fig. 4?

> *Correction.*
> On Fig. 4(b), one can see that the fringes inside the magnet are shifted from the ones outside of the magnet. It shows that even if the magnetic field of the magnet is confined to the magnet, it influences the electrons passing through the hole of the magnet. Fig. 4(c) shows a schematic of the electronic wavefront (retarted inside the magnet).

13. Which potential problems do the authors discuss?

> *Correction.*
> Field-leakage (the main problem of the solenoid geometry) and electron penetrability inside the magnet.

14. Why did the authors choose a toroidal magnet? Give at least two reasons.

> *Correction.*
> To avoid the field leakage. To allow electrons to penetrate the magnet in order to get the smoothly varying phase difference, and not only the global phase shift.

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PHYSICAL REVIEW LETTERS

VOLUME 48 24 MAY 1982 NUMBER 21

Observation of Aharonov-Bohm Effect by Electron Holography

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(Received 16 February 1982)

In this experiment, an electron- and optical-holographic technique is employed with small toroidal ferromagnets each forming a magnetic-flux closure. The holographic interferometry proves that a phase difference between two electron beams having passed through the field-free regions agrees well with the fundamental relation known as the Aharonov-Bohm effect. It is also confirmed from the same hologram that flux leakage from the toroids does not affect the conclusion.

PACS numbers: 03.65.Bz, 41.80.Dd, 42.40.Mg

The existence of the Aharonov-Bohm effect[1] (AB effect) has recently been questioned by Bocchieri *et al.*[2] and Roy.[3] The AB effect states that a phase difference between two electron beams is produced proportional to the enclosed magnetic flux, even if they never touch the magnetic field. Bocchieri *et al.* asserted that the AB effect is purely of mathematical origin. Experiments in the past[4] were also questioned from the standpoint that electrons were affected by inevitable leakage magnetic fields from finite whiskers or solenoids used in these experiments.[4] Although these assertions have since then been disputed theoretically by many authors,[5] the controversy has still not fully abated.[6]

Our experiment employs electron holography.[7] In order to avoid the questioned leakage effects,[8] tiny toroidal magnets[9] were used instead of whiskers or solenoids to make complete flux circuits. Furthermore, a new method of holographic interference microscopy[10] was employed, both to obtain contour maps of the electron phase and to detect quantitatively the amount of leakage that might have, by some chance, come from the magnets.

The toroidal magnets were prepared in the following way. Permalloy thin films (80% Ni and 20% Fe) were prepared by vacuum evaporation. The substrate was a glass plate covered with an evaporated thin film of NaCl. Permalloy toroids of various sizes were formed by means of electron-beam lithography. These toroids were floated off on a water surface, and applied to thin carbon films approximately 100 Å thick.

An electron-microscopic image and an underfocused Lorentz micrograph of such a toroidal magnet are shown in Figs. 1(a) and 1(b). The Lorentz micrograph shows that magnetization is closed within the magnet. This is the case with most toroids presumably as a result of the shape effect.

Off-axis electron homograms of the toroidal magnets were formed in a 100-kV field-emission electron microscope.[11] The schematic diagram for hologram formation is shown in Fig. 2. A toroidal magnet was illuminated with a collimated electron beam. Its demagnified image (magnification ~ $\frac{1}{2}$) was formed through both ob-

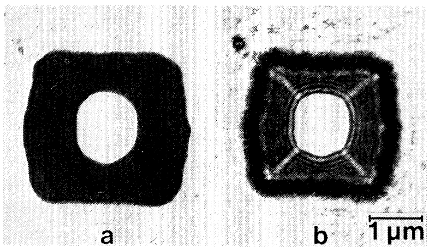

FIG. 1. Toroidal magnet. (a) Electron-microscopic image. (b) Lorentz micrograph.

jective and intermediate lenses. A reference beam was projected on the image plane by an electron biprism[12] to form the image hologram. Final magnification at the recorded hologram was 2 000.

Reconstruction was performed in the optical system shown in Fig. 3. A collimated laser beam from a He-Ne laser was split into two beams by beam splitter *A*. One beam illuminated the electron hologram to reconstruct the image, which was focused again by lenses *E* and *F* on the observation plane. The other beam (comparison beam) from the splitter was superposed on the observation plane to form the interference image. The advantage of the holography technique is that it makes it possible to obtain phase-amplified interference images.[13]

Interference micrographs for the toroidal magnet (Fig. 1) are shown in Fig. 4. The phase contour map of an electron beam transmitted through a magnet, shown in Fig. 4(a), was obtained with the comparison beam parallel to the object beam. It cannot be determined from the contour map whether the wave front of the object beam is advanced or retarded. Therefore, interferogram (b) was taken with a tilted comparison beam to determine this. The wave front obtained is schematically shown in Fig. 4(c).

The photographs reveal that a phase difference really exists between two electron beams that have passed through the inner and outer spaces of a toroidal magnet, where there were no magnetic fields in those spaces. In addition, the phase difference of 5.5λ, measured from the interference micrograph, agrees with the theoretical value of 6.0λ to 20%. This is estimated from data where 4π times magnetization was 9500 ± 500 Oe, film thickness was 400 ± 30 Å, and toroid width was 6400 ± 500 Å. Phase shifts at the magnet edges are partly due to the refraction effect.[13] However, this effect can be ignored in the present estimation, because the effect of

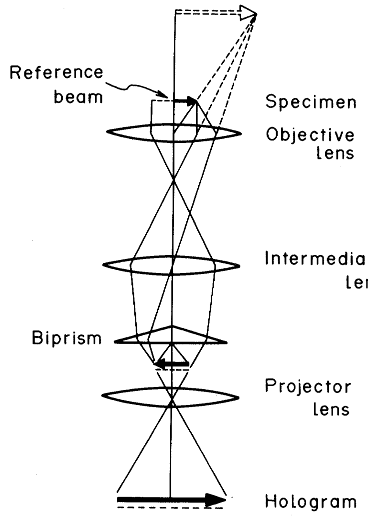

FIG. 2. Schematic arrangement for electron hologram formation.

the phase shift made at the outer edge is cancelled at the inner edge. In addition the shift value itself is smaller than one λ, as is also explained from the thickness and inner potential (~20 V) of the sample.

Another example with a slightly larger magnet on the same carbon film is shown in Fig. 5. In contrast to the previous example, the phase here is retarded in the inner space of the magnet. Correspondingly, the magnetization direction is counterclockwise. The number of contour lines, i.e., the phase difference, is increased compared with that in Fig. 4. This is in proportion to the magnet width since film thicknesses are the same. The deviation from the proportional relation was measured to be less than 10% for

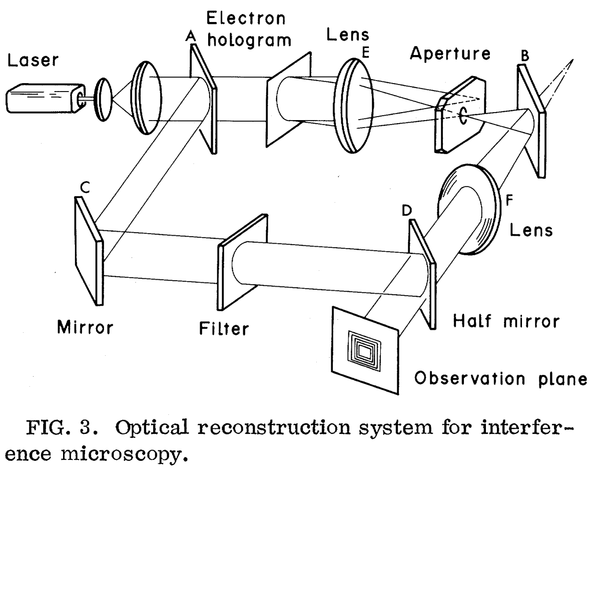

FIG. 3. Optical reconstruction system for interference microscopy.

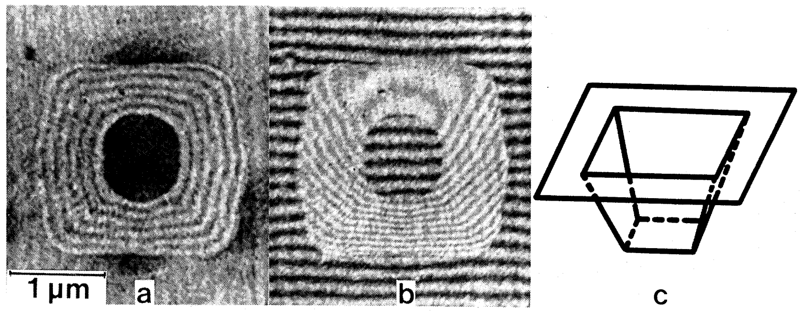

FIG. 4. Interference micrographs of the toroidal magnet shown in Fig. 1. (a) Contour map of electron phase. (b) Interferogram of electron phase. (c) Schematic form of the wave front.

various toroidal sizes.

These experimental results verify the existence of the AB effect. Quantitative agreement[14] is achieved with the fundamental AB effect relation.

Leakage-field effects were confined to be sufficiently small in the cases of Figs. 4 and 5. Contour lines in interference micrographs were verified to follow magnetic lines of force as viewed along the direction of the electron beam.[15] Therefore, contour lines must exit from the toroid if magnetic fields are leaking from the magnet. An example of field leakage is shown in Fig. 6. Leakage fields do not show up in the Lorentz micrograph, Fig. 6(a), but can be clearly observed in the interference micrograph, Fig. 6(b). The magnetic flux between two adjacent contour lines is equal to a constant, h/e, irrespective of electron energy. It can be concluded from the contour maps shown in Figs. 4(a) and 5(a) that the leakage flux was less than h/e and that the resultant phase change is too small to conceal the AB effect.

In this experimental arrangement, the electron beam partly touched and even penetrated the magnet. This point is open to criticisms, but our argument for this is as follows. In the present experiment, the shape of a magnetic sample is reproduced as a clear image on the interferogram. Consequently, the part of the beam transmitted through the magnetic flux in the sample does not contribute to points outside the sample image. The beams reaching these points must have felt only the magnetic vector potential, if any.

It was for the measurement of the phase difference by tracing the interference fringes that the penetrable toroidal magnets were adopted in our experiment. This is an advantage of our experiment over former experiments.[4] If the fringes on the images of the toroids are not observed, the phase difference is determined by only a fraction of a wavelength unit.[16]

The different electron energy causes an appreciable change in electron penetrability, but no change in phase difference. This fact was confirmed at 80, 100, and 125 kV. If there were an essential difference between an absolutely inaccessible field and a negligibly accessible field,[6] then the AB effect could be neither confirmed nor denied experimentally.

Regardless of the strength of penetrability, our experimental results of the interference

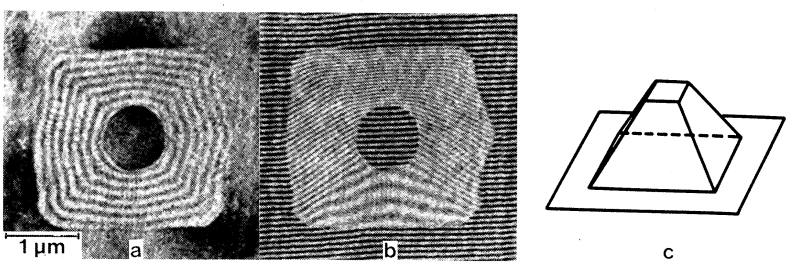

FIG. 5. Interference micrographs of magnet having a magnetization direction opposite to that in Fig. 4. (a) Contour map. (b) Interferogram. (c) Schematic form of the wave front.

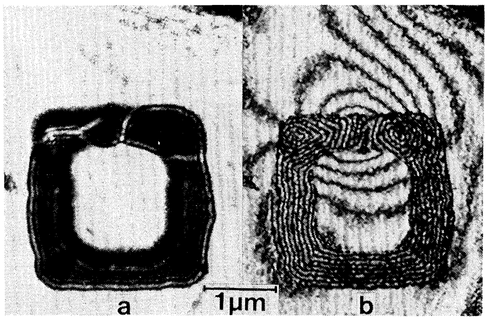

FIG. 6. Example of leakage magnetic fields. (a) Lorentz micrograph. (b) Interference micrograph. (Phase amplification by factor two.)

pattern, e.g., Fig. 4(b), can be fully explained with the Stokes vector potential. This cannot be expected if vector potential is zero everywhere outside the toroid, as Bocchieri *et al.* proposed in case of a solenoid.[2]

We are very grateful to Professor C. N. Yang of State University of New York, Stony Brook, for useful discussions at the planning stage. We would like to express our sincere thanks to Dr. T. Doi for his support and encouragement of such a fundamental research work. We also gratefully acknowledge the valuable advice and stimulation given by Professor R. Uyeda of Meijo University and by Dr. H. Okano, Dr. M. Kudo, Dr. T. Komoda, and Dr. F. Nagata of our laboratory.

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